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Streamwise Computation of Two-Dimensional Incompressible Potential Flows

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A new approach to calculate two-dimensional plane and axisymmetric incompressible potential flows is presented. The dependent variables of the new approach are the streamwise velocity, along a set of chosen streamlines, and the coordinates of the chosen streamlines in the cross-stream plane. Thus, the method generates directly the streamline pattern for a given flow. The method is particularly well suited for computing flow through complex configurations. © 1985 Academic Press, Inc.

I. INTRODUCTION

Use of the stream function or the streamline coordinates for the computation of flow fields is well known. For example, in the method of hodograph (e.g., Oswatitsch [1]) equations for the potential flow are formulated using the streamlines and the orthogonal trajectories as the dependent variables, and the velocity components as the independent variables. Uchida and Yasuhara [2] and Ishii [3] have calculated axisymmetric potential flows using this set of variables. Another interesting study that makes use of the streamline coordinates is the work of Pearson [4] on the computation of isentropic flows. In this study one starts with a guessed streamline geometry for the given flow and then calculates the density to satisfy the Bernoulli's equation and the guessed streamline geometry. With the density thus known, the streamline geometry is next corrected to satisfy the individual momentum equations for the velocity components. The streamline geometry, thus obtained, provides the starting point of the next cycle of iteration.

Over the past few years, we have been developing a new approach to the use of streamlines for the computation of two- and three-dimensional flow fields. The dependent variables of this approach are the streamwise (not axial) velocity along a set of chosen streamlines, and the coordinates of the chosen streamlines in the cross-stream plane. Let the main flow direction be along x, and (y, z) be the coordinates of the cross-stream plane. Further, let subscript s represent a particular streamline. The dependent variables of the new approach are: $U_s(x, t)$, streamwise velocity along the streamline s; and $Y_s(x, t)$ and $Z_s(x, t)$, coordinates of the

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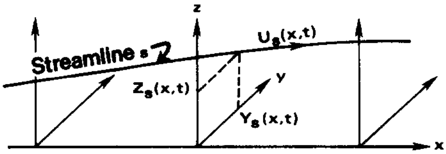

FIG. 1. The dependent variables: U_s, the streamwise velocity; Y_s and Z_s, the streamline coordinates in the cross-stream plane.

streamline s in the cross-stream plane (see Fig. 1). The independent variables are t, the time; x, the distance along the main flow direction (distance along the duct axis for internal flows); and the initial coordinates of the streamline s.

The differential equation for U_s is simply the equation of motion of a fluid element moving along a streamline, i.e.,

$$\frac{DU_s}{Dt}=f_s, \tag{1}$$

where f_s is the force per unit mass acting on the fluid element in the streamwise direction. By expressing f_s in terms of f_x, f_y, and f_z, the rectangular components of f, Eq. (1) can be written as

$$\frac{DU_s}{Dt}=\frac{f_x+Y_s'f_y+Z_s'f_z}{\sqrt{(1+Y_s'^2+Z_s'^2)}}, \tag{2}$$

where the prime indicates derivatives with respect to x. To derive the differential equation for Y_s, project the motion of the fluid element onto the x–y plane as shown in Fig. 2. In Fig. 2, s_{xy} represents the projection of the streamline s on the x–y plane, and f_n is normal to s_{xy}. By applying Newton's second law of motion in the direction of f_n, one obtains

$$\frac{1}{R_{xy}}=\frac{f_n}{U_{sxy}^2}, \tag{3}$$

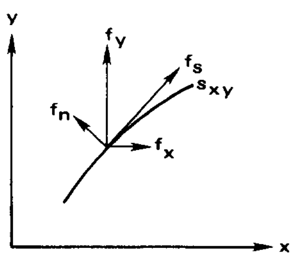

FIG. 2. Projection of the streamline s on the (x, y) plane.

where R_{xy} is the radius of curvature of s_{xy}, and $U_{s_{xy}}$ is the projection of U_s on the x–y plane. By expressing R_{xy} in terms of Y'_s and Y''_s, $U_{s_{xy}}$ in terms of U_s, and f_n in terms of f_x and f_y, Eq. (3) can be rewritten as

$$\frac{Y''_s}{(1+Y'^2_s)^{3/2}}=\frac{(1+Y'^2_s+Z'^2_s)}{(1+Y'^2_s)\,U^2_s}\left[\frac{f_y}{\sqrt{(1+Y'^2_s)}}-\frac{Y'_s f_x}{\sqrt{(1+Y'^2_s)}}\right]$$

or more simply as,

$$Y''_s=(1+Y'^2_s+Z'^2_s)(f_y-Y'_s f_x)/U^2_s. \tag{4}$$

Similarly one obtains for Z,

$$Z''_s=(1+Y'^2_s+Z'^2_s)(f_z-Z'_s f_x)/U^2_s. \tag{5}$$

These equations were solved for two-dimensional flows in Ref. [5], and for three-dimensional flows in Ref. [6], with the approximation that the flows were "parabolic" (boundary layer) type flows. For such flows the streamline curvature, associated with the deflection of the streamlines caused by the continuity equation, is assumed negligible and Eqs. (4) and (5) simply yield the familiar result that the pressure field in the cross-stream plane is uniform. Also, Y'_s and Z'_s in Eq. (2) were neglected in comparison with unity. In the present paper, a solution of these equations for steady, incompressible, two-dimensional plane and axisymmetric potential flows is presented.

II. Two-Dimensional Potential Flow

Plane Flows

For steady, incompressible, and inviscid flow, Eq. (2) when integrated yields the Bernoulli's equation,

$$\frac{U^2_s}{2}+\frac{p_s}{\rho}=C, \tag{6}$$

where p_s and ρ are the pressure and density, respectively, along the streamline s, and C is a constant. For uniform inlet flow conditions for internal flows, and for uniform flow far away from the obstacle for external flows, C will be the same for all the streamlines. In the present study it is assumed that C is same for all the streamlines. For the sake of brevity, the subscript s in Y_s has been omitted from here on, and angle α, defined by

$$\tan\alpha=Y',$$

has been introduced.

In the present case the force f_n in Eq. (3), being solely due to the pressure gradients in the cross-stream plane, is easier to calculate when expressed in terms of f_y and f_s, its components along y and s, respectively (see Fig. 2). One thus obtains.

$$Y'' \cos^3 \alpha = -\frac{1}{\rho U_s^2}\left[\frac{1}{\cos \alpha}\frac{\partial p_s}{\partial y} - \tan \alpha \frac{\partial p_s}{\partial s}\right]. \tag{7}$$

Further the gradient of p_s with respect to s is expressed as

$$\frac{\partial p_s}{\partial s} = \cos \alpha \frac{\partial p_s}{\partial x}. \tag{8}$$

Next the pressure gradients in Eq. (7) are expressed in terms of the velocity gradients by the use of Eq. (6) to obtain

$$Y'' \cos^3 \alpha = \frac{1}{U_s}\left[\frac{1}{\cos \alpha}\frac{\partial U_s}{\partial y} - \sin \alpha \frac{\partial U_s}{\partial x}\right]. \tag{9}$$

Let ψ denote the volume flow rate between the streamline s and some reference streamline s_0. The continuity equation can now be expressed as

$$\int_{Y_{s_0}}^{Y_s} U_s \cos \alpha \, dy = \psi$$

or in the differential form as

$$U_s = \frac{1}{Y_\psi \cos \alpha}, \tag{10}$$

where the subscript ψ represents derivative with respect to ψ. From Eq. (10) one obtains for the velocity derivatives;

$$\frac{\partial U_s}{\partial y} = -\frac{Y_{\psi\psi}}{\cos \alpha Y_\psi^3} + \frac{Y' Y'_\psi \cos \alpha}{Y_\psi^2}, \tag{11}$$

$$\frac{\partial U_s}{\partial x} = -\frac{Y'_\psi}{\cos \alpha Y_\psi^2} + \frac{Y' Y'' \cos \alpha}{Y_\psi}. \tag{12}$$

By substituting Eqs. (11) and (12) into Eq. (9), restoring α in terms of Y', and some rearrangement, one obtains

$$Y_{xx} Y_\psi^2 + Y_{\psi\psi}(1 + Y_x^2) - 2 Y_x Y_\psi Y_{x\psi} = 0, \tag{13}$$

where, for the sake of uniformity in the final equation, the subscript x has been used to denote the derivatives with respect to x. In Eq. (13) Y, the streamline coordinate, appears as a function of x and ψ. The dependence on ψ represents the dependence of Y on its starting ordinate $Y(x_0)$. Equation (13) can also be obtained from the Laplace equation for ψ by transformation of the variables as shown in the Appendix.

Equation (13) along with Eq. (10) constitute the basic equations of the present approach to compute plane potential flows. Equation (10) determines U_s, the streamwise velocity along streamline s, and Eq. (13) determines Y, the coordinate of the streamline s.

Axisymmetric Flows

Equations (6), (7), and (9) are equally valid for flows which are symmetric with respect to an axis. The variable Y, now represents the radial distance of the streamline s from the axis of symmetry. The continuity equation, however, now takes the form

$$U_s = \frac{1}{2\pi Y Y_\psi \cos\alpha}. \tag{14}$$

By using Eq. (14) to evaluate the velocity gradients in Eq. (9), and then proceeding as in the case of plane flows, one obtains the following equation for the streamline ordinate:

$$Y_{xx} Y_\psi^2 + Y_{\psi\psi}(1 + Y_x^2) - 2 Y_x Y_\psi Y_{x\psi} + Y_\psi^2 / Y = 0. \tag{15}$$

III. Finite Difference Approximation and Sample Computations

Let the subscript j denote a particular streamline, and the subscript i distance along x. Thus, $Y_{i,j}$ represents y coordinate of the streamline j at $x = (i-1)\,\Delta x$. Finite difference approximation to Eq. (13) is obtained by using center differencing [7] for all the derivatives; thus

$$Y_x = \frac{(Y_{i+1,j} - Y_{i-1,j})}{2\Delta x} \equiv a,$$

$$Y_{xx} = \frac{(Y_{i+1,j} - 2Y_{i,j} + Y_{i-1,j})}{\Delta x^2},$$

$$Y_\psi = \frac{(Y_{i,j+1} - Y_{i,j-1})}{(d_{j+1} + d_j)} \equiv b,$$

where for brevity we have introduced d_j, the volume flow rate between the streamlines j and $j-1$, defined by the relation

$$d_j = \psi_j - \psi_{j-1},$$

$$Y_{\psi\psi} = \frac{Y_{i,j+1} d_j - Y_{i,j}(d_{j+1} + d_j) + Y_{i,j-1} d_{j+1}}{0.5(d_{j+1} + d_j)\, d_{j+1} d_j},$$

$$Y_{x\psi} = \frac{Y_{i+1,j+1} - Y_{i+1,j-1} - Y_{i-1,j+1} + Y_{i-1,j-1}}{2\Delta x(d_{j+1} + d_j)}. \tag{16}$$

By substituting Eqs. (16), and some rearrangement, one obtains

$$Y_{i,j}=\left[\frac{2b^2}{\Delta x^2}+\frac{2(1+a^2)}{d_j d_{j+1}}\right]^{-1}\left[\frac{b^2}{\Delta x^2}(Y_{i+1,j}+Y_{i-1,j})\right.$$
$$\left.+\frac{(Y_{i,j+1}d_j+Y_{i,j-1}d_{j+1})(1+a^2)}{0.5(d_{j+1}+d_j)\,d_j d_{j+1}}-2abY_{x\psi}\right], \tag{17}$$

where a, b, d, and $Y_{x\psi}$ are given in Eqs. (16).

Sample computations were carried out for flow past a semi-infinite plane cylinder of width $\pi/2$ placed symmetrically in a channel of width π. The exact solution for this problem is given by [8]

$$\psi=y+\frac{1}{2}\left\{y-\tan^{-1}\left[\frac{\tan y}{\tanh x}\right]\right\}. \tag{18}$$

Flow is along the x axis and the stagnation point (tip of the cylinder), x_q, is $\tanh^{-1}\frac{1}{3}$ to the left of $x=0$. Since the flow is symmetric with respect to the mid-plane, the computations were carried out only in the upper half of the channel. The grid and the boundary values of $Y_{i,j}$ used in the computations were as follows:

No. of streamlines:	11	
Integration step:	$\Delta x=x_q/3=\tanh^{-1}(\frac{1}{3})/3$	
Upper boundary:	$Y_{i,11}=\pi/2$	
Lower boundary:	$Y_{i,1}=0$	for $x\leqslant x_q$
	$=y$	given by Eq. (18) with $\psi=0$, for $x>x_q$
Channel inlet:	$x=-23\Delta x$;	$Y_{i,j}=(j-1)\,\pi/20$
Channel exit:	$x=20\Delta x$;	$Y_{44,j}=(j-1)(Y_{44,11}-Y_{44,1})/10$

For this case all the d_j are equal and the corresponding finite difference equation for $Y_{i,j}$ is simply obtained from Eq. (17) by deleting all the d_j. The resulting equation was solved by the method of successive displacement (Liebman method). Values of $Y_{i,j}$ were displaced until the maximum difference between the new values and the old values was less than 0.0001. The computations, carried out on IBM 370/4341 operating under DOS operating system, took 2.8 s of CPU time.

Results of the computations are shown in Fig. 3. The exact streamline pattern, given by Eq. (18), is too close to the computed streamlines, except for the streamline $j=2$ ($\psi=\pi/20$), to be shown in the figure. The exact streamline for $j=2$ is shown in Fig. 3 by a broken line where it differs from the corresponding computed streamline.

In the computations just described, the starting values assigned to $Y_{i,j}$ at the interior grid points were such that the streamlines were spaced equally between the

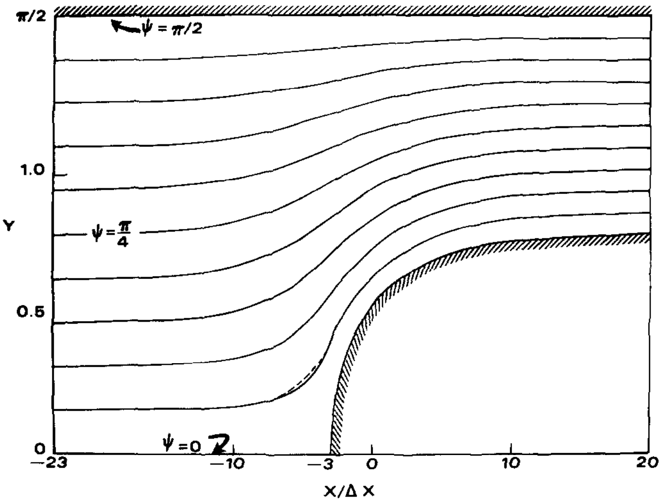

FIG. 3. Streamline pattern for flow past a semi-infinite cylinder of width $\pi/2$ placed symmetrically in a channel of width π. Solid lines represent the computed streamline pattern, and the broken line the exact solution where it differs from the computed one.

top and the bottom wall for all x. To check the sensitivity of the algorithm to the starting values of $Y_{i,j}$, computations were repeated by assigning a starting value of zero to $Y_{i,j}$ at all the interior grid points. The results thus obtained were indistinguishable from the earlier results; however, this time the computations took 5.8 s of CPU time.

IV. Concluding Remarks

A new approach, that uses the streamline coordinates as part of the dependent variables, has been presented to compute potential flows. The basic equation of this approach, Eq. (13), is nonlinear; this is in contrast to the linear equation, Laplace equation for the stream function, used in the conventional approach to solve potential flow problems. However, because of the nature of the dependent variables employed, finite differencing of Eq. (13) does not require any special treatment in the neighborhood of the curved boundaries. The same finite difference approximation to Eq. (13), for example, Eq. (17), is equally applicable everywhere.

In calculating flows past solid boundaries using Eulerian flow description and Cartesian coordinate system, one encounters difficulty when the mesh points of the coordinate system do not fall naturally on the solid boundaries. To overcome this difficulty, arising from the incompatibility of the Cartesian computational mesh and the boundaries, many authors (see, for example, Roberts and Forestor [9]) have in the recent years employed body fitted coordinate systems for computting flow through ducts of arbitrary cross sections. The present approach calculates the streamline pattern, a body fitted coordinate system, as part of its dependent variables; thus, it seems quite well suited for computing flow through complex geometries.

APPENDIX

In this Appendix, Eq. (13) is derived from the Laplace equation for the stream function ψ,

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0, \tag{A1}$$

by transformation of the variables. Let

$$\eta=\eta(x, y) \qquad \text{and} \qquad \xi=\xi(x, y)=x. \tag{A2}$$

By using the standard technique for evaluating the second derivatives of the new variables with respect to the old variables (see, for example, Ref. [10]) one obtains for the transformation (A2),

$$\eta_{xx}=\frac{1}{y_\eta}\left[2y_{x\eta}\frac{y_x}{y_\eta}-y_{xx}-y_{\eta\eta}\left(\frac{y_x}{y_\eta}\right)^2\right], \tag{A3}$$

$$\eta_{yy}=-\frac{y_{\eta\eta}}{y_\eta^3}. \tag{A4}$$

By letting η be ψ, substituting Eqs. (A3) and (A4) into Eq. (A1), and some rearrangement, one obtains Eq. (13).

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